# Congruence module.

Reference: "Congruence modules in higher codimension and zeta lines in Galois cohomology"
"Congruence modules and the Wiles-Lenstra-Diamond numerical criterion in higher codimension."

## § Notation.

$\mathcal{O}$ : complete discrete valuation ring
$\varpi$ : uniformizer
$A$ : complete local $\mathcal{O}$-algebra
$M$ : f.g. $A$ module
$\lambda : A \to \mathcal{O}$ map of $\mathcal{O}$ algebra.

$P_\lambda := \ker\lambda \qquad c := \text{height } P_\lambda = \dim A_{P_\lambda}$.

$F_\lambda^i(M) := \operatorname{Ext}_A^i(\mathcal{O}, M)^{tf}$

$\Psi_\lambda(M) = \operatorname{coker}(\underbrace{\operatorname{Ext}_A^c(\mathcal{O}, M)}_{\text{是 f.g. } \mathcal{O}\text{-module}}{}^{tf} \longrightarrow \operatorname{Ext}_A^c(\mathcal{O}, M/P_\lambda M)^{tf})$

congruence module.

$\Phi_\lambda(A) := \operatorname{tors}(P_\lambda/P_\lambda^2)$ torsion part of cotangent module.

Thm: TFAE

① The local ring $A_{P_\lambda}$ is regular
② The rank of the $\mathcal{O}$-module $P_\lambda/P_\lambda^2$ is $c=$ height $P_\lambda$.
③ The $\mathcal{O}$-module $\Phi_\lambda(A)$ is torsion
④ The $\mathcal{O}$-module $\Psi_\lambda(M)$ is torsion for each f.g. $A$ module $M$

When the condition holds, the $\Theta$-module $\Psi_\lambda(A)$ is cyclic.

$\mathcal{C}_\Theta$ : the category whose objects are pairs $(A,\lambda)$ satisfying above equivalent condition

$\mathcal{C}_\Theta(C)$: subcategory of $\mathcal{C}_\Theta$ consists of pairs $(A,\lambda)$ s.t $ht(P_\lambda)=C$.

Lemma. For any $(A,\lambda)\in\mathcal{C}_\Theta(C)$ and f.g. A-module $M$.

$$\begin{array}{ccc} F_\lambda^C(M) & \longrightarrow & F_\lambda^C(M/P_\lambda M) \\ \| & & \| \\ \operatorname{Ext}_A^C(\Theta, M)^{tf} & & \operatorname{Ext}_A^C(\Theta, M/P_\lambda M) \end{array}$$ is injective

Thm. For $(A,\lambda)\in\mathcal{C}_\Theta(C)$, the local ring $A$ is regular if and only if $\Phi_\lambda(A)=0$ if and only if $\Psi_\lambda(A)=0$

## § structure of $F_A^*(\Theta)$

The Ext-algebra $\operatorname{Ext}_A^*(\Theta,\Theta)$ as graded $\Theta$-algebra can be highly non-commutative and infinite. However, its torsion free quotient $F_\lambda^*(\Theta)$ has simple structure, which is an exterior algebra generated by its degree one components.

$$F_\lambda^1(\Theta)\cong \operatorname{Hom}_\Theta(P_\lambda/P_\lambda^2,\Theta)$$

$$F_\lambda^C(\Theta)=\Lambda^C \operatorname{Hom}_\Theta(P_\lambda/P_\lambda^2,\Theta).$$

Lemma: $F_\lambda^1(\Theta)=\operatorname{Ext}_A^1(\Theta,\Theta)\cong \operatorname{Hom}_\Theta(P_\lambda/P_\lambda^2,\Theta)$

Proof: we have

$$0 \to P_\lambda \to A \to \mathcal{O} \to 0$$

Applying $\mathrm{Hom}_A(-, M)$

$$\underset{\substack{\wr\wr \\ M}}{\mathrm{Hom}_A(A, M)} \to \mathrm{Hom}_A(P_\lambda, M) \to \mathrm{Ext}^1_A(\mathcal{O}, M) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^1_A(A, M)}$$

$$\mathrm{Ext}^1_A(\mathcal{O}, M) = \mathrm{Coker}(M \longrightarrow \mathrm{Hom}_A(P_\lambda, M))$$

Take $M = \mathcal{O}$

$$\underset{\substack{\| \\ \mathcal{O}}}{\mathrm{Hom}_A(A, \mathcal{O})} \longrightarrow \mathrm{Hom}_A(P_\lambda, \mathcal{O}) \text{ is zero map}$$

$$\mathrm{Ext}^1_A(\mathcal{O}, \mathcal{O}) = \mathrm{Hom}_A(P_\lambda, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(P_\lambda/P_\lambda^2, \mathcal{O})$$

which is already torsion free.

## § Freeness criterion.

For any $A$-module $X$, we have künneth map.

$$\mathrm{Ext}^c_A(\mathcal{O}, X) \otimes_{\mathcal{O}} (M/PM) \simeq \mathrm{Ext}^c_A(\mathcal{O}, X) \otimes_A M \longrightarrow \mathrm{Ext}^c_A(\mathcal{O}, X \otimes_A M)$$

This is functorial in $X$.

Take $X = A$ and $\mathcal{O}$ and torsion free quotient.

$$
\begin{tikzcd}
\mathrm{Ext}_A^c(\mathcal{O},A)^{tf} \otimes_{\mathcal{O}} (M/P_\lambda M)^{tf} \arrow[r, hook] \arrow[d] & \mathrm{Ext}_A^c(\mathcal{O},\mathcal{O})^{tf} \otimes_{\mathcal{O}} (M/P_\lambda M)^{tf} \arrow[d, "\wr"] \\
\mathrm{Ext}_A^c(\mathcal{O},M)^{tf} \arrow[r, hook] & \mathrm{Ext}_A^c(\mathcal{O}, M/P_\lambda M)^{tf}
\end{tikzcd}
$$

The diagram above induces a natural surjective map of $\mathcal{O}$-modules

$$a_\lambda(M): \ \Psi_\lambda(A)^n \twoheadrightarrow \Psi_\lambda(M) \qquad n = \mathrm{rank}_\lambda(M) = \mathrm{rank}_{A_p}(M_p)$$

In particular, there is an equality

$$\mathrm{length}_{\mathcal{O}} \Psi_\lambda(M) = n \cdot \mathrm{length}\, \Psi_\lambda(A) - \mathrm{length}_{\mathcal{O}} \ker a_\lambda(M)$$

Thm: 2.19. With notation above. Further assume $A$ Gorenstein and $M$ is maximal Cohen-Macaulay.

$$\mathrm{Length}_{\mathcal{O}} \Psi_\lambda(M) = n \cdot \mathrm{length}_{\mathcal{O}} \Psi_\lambda(A) \Longleftrightarrow M \simeq A^n \oplus W \text{ and } W_{P_\lambda} = 0 \text{ as } A\text{-modules}$$

Def.

Wiles defect

$$\delta_\lambda(M) = \mathrm{rank}_\lambda(M) \cdot \mathrm{length}_{\mathcal{O}} \Phi(A) - \mathrm{length}_{\mathcal{O}} \Psi_\lambda(M)$$

plug in above formula, we have

$$\delta_\lambda(M) = \mathrm{rank}_\lambda(M) \cdot \delta_\lambda(A) + \mathrm{length}_{\mathcal{O}} \ker(a_\lambda(M))$$

Which tells us. $\delta_\lambda(M) \geq 0$ for all $M$ if and only if $\delta_\lambda(A) \geq 0$

Thm: When $(A, \delta) \in C_\Theta(c)$ with depth $A \geq c+1$ one has $\delta_\lambda(A) \geq 0$
The equality holds if and only if $A$ is complete intersection.

Thm When $\operatorname{depth}_A M \geq c+1$ and $M_{p_\lambda} \neq 0$, one has $\delta_\lambda(M) \geq 0$
The equality holds if and only if. $A$ is complete intersection
and $M \cong A^n \oplus W$ and $W_{p_\lambda} = 0$